Constraining nonextensive statistics with plasma oscillation data

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(Dated: February 3, 2008)

We discuss experimental constraints on the free parameter of the nonextensive kinetic theory from measurements of the thermal dispersion relation in a collisionless plasma. For electrostatic plane-wave propagation, we show through a statistical analysis that a good agreement between theory and experiment is possible if the allowed values of the q-parameter are restricted by $q = 0.77 \pm 0.03$ at 95% confidence level (or equivalently, $2 - q = 1.23$, in the largely adopted convention for the entropy index q). Such a result rules out (by a large statistical margin) the standard Bohm-Gross dispersion relation which is derived assuming that the stationary Maxwellian distribution ($q = 1$) is the unperturbed solution.

PACS numbers: 52.35.Fp, 05.45.-a, 05.20.-y, 05.90.+m

It is usually assumed that the particle velocity distribution in a collisionless thermal plasma eventually relaxes to the standard Maxwellian curve [1, 2]. However, the possibility of a non-Gaussian behavior was established since the earlier experiments in plasma physics [3]. More recently, Langmuir probe measurements have also suggested that the isotropic component of the electron distribution resulting from inverse bremsstrahlung absorption in plasma is well described by a non Maxwellian distribution [4]. In principle, such results may be explained by a stationary distribution that emerges naturally from the nonextensive statistical formalism proposed by Tsallis [5]. Such an approach generalizes the standard Boltzmann-Gibbs formalism through a new analytic form for the entropy $S_q = k_{\rm B}(1 - \sum_i p_i^q)/(q-1)$, where $k_{\rm B}$ is the standard Boltzmann constant, p_i is the probability of the ith microstate, and q is a parameter quantifying the degree of nonextensivity. This expression has been introduced in order to extend the applicability of statistical mechanics to system with long range interactions and has the standard Gibbs-Jaynes-Shannon entropy as a particular limiting case ($q = 1$). The associated nonextensive kinetic theory has also been considered in many different physical scenarios, ranging from astrophysics to plasma physics (see [6] for a regularly updated bibliography). For example, it has been successfully applied to two-dimensional Euler and drift turbulence in a pure-electron plasma column [7], Lévy-type anomalous diffusion [8], anomalous relaxation through electron-phonon interaction [9], ferrofluid-like systems [10], plasma oscillations [11], the solar neutrino problem [12], astrophysical systems [13], among others. This new formalism has also shown to be endowed with several interesting mathematical properties [14], with the main theorems of the standard statistics admitting suitable generalizations [15]. More recently, some authors have argued that the value of the nonextensive parameter is predictable for some simple physical systems, as for instance, in the case of logistic maps. Their analyzes take into account the relation between sensitivity to initial conditions and relaxation with the q-generalized Lyapunov coefficient usually computed in terms of the map parameters [16]. Theoretically, the situation is not so neat for long-range Hamiltonian systems as it is for maps. The present belief is that systems with short range interactions obeys the Boltzmann-Gibbs statistics and exponential relaxation dictates the approach to equilibrium. However, for more complex systems (including the case of plasmas), it is not possible to determine apriori how the relaxation to a stationary state takes place. In some cases, due to a formation of correlated clusters in N-Body systems, a preequilibrium stage may happens much earlier than the final Boltzmann-Gibbs equilibrium in such a way that the overall effect can be described by a power-law Tsallis distribution [17]. The interesting feature of this power-law function is that many models are analytically tractable so that a detailed comparison with the Boltzmann-Gibbs approach is immediate. In this letter, we discuss new constraints on the q-parameter associated to measurements of the dispersion relations for electrostatic plasma oscillations. Through a statistical analysis we estimate the value of the nonextensive

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parameter from the existing experimental data. More precisely, by analyzing the data set for thermal dispersion relation taken from Van Hoven [18] we argue that a reasonable agreement between theory and experiment is possible for values of the nonextensive parameter restricted by $q = 0.77 \pm 0.03$ at 95% confidence level. Some years ago, it was shown how Maxwell's derivation [19] to the stationary velocity distribution function could be extended to the nonextensive domain [20, 21]. In the one-dimensional case, the new distribution reads

$$f_0(v_x) = A_q \left[1 - (q-1)\frac{mv_x^2}{2k_BT}\right]^{\frac{1}{q-1}}. \tag{1}$$

The above expression has been determined from two simple requirements: (i) the isotropy of the velocity space, and (ii) a nonextensive generalization of the Maxwell factorizability condition, that is, $F(v) \neq f(v_x)f(v_y)f(v_z)$[20]. The quantity A_q in the above equation denotes the q-normalization constant and can be written, respectively, for the intervals $0 < q \leq 1$ and $q > 1$, by

$$A_{0<q\leq 1} = \frac{n\Gamma(\alpha)}{\Gamma(\alpha - \frac{1}{2})}\sqrt{\frac{m(1-q)}{2\pi k_BT}} \tag{2}$$

and

$$A_{q>1} = n\left(\frac{1+q}{2}\right)\frac{\Gamma(\frac{1}{2}+\alpha)}{\Gamma(\alpha)}\sqrt{\frac{m(q-1)}{2\pi k_BT}}, \tag{3}$$

where $\alpha = 1/(q-1)$ is a dimensionless number, n is the particle number density, m is the mass of the particle and T is the temperature. For $q > 1$, the distribution (1) exhibits a thermal cutoff on the maximum value allowed for the velocity of the particles. However, we notice that the above velocity distribution is parameterized with the nonextensive parameter shifted by $q \rightarrow 2-q$. This means that our thermal cutoff at $q > 1$ is equivalent to the cutoff condition, $q < 1$, usually adopted by other authors. As one may check, in the extensive limit, $q = 1$, the above distribution function reduce to the standard Gaussian form [11, 19, 20]. Another interesting result leading to a coherent nonextensive kinetic theory is related to the transport equation. In this context, the generalized Boltzmann's equation reads [21]

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = C_q(f), \tag{4}$$

where the left-hand-side is the total time derivative of f while C_q is the q-nonextensive collisional integral term responsible by changes in the distribution function f. In this approach, the stationary velocity q-distribution was obtained from a slight generalization of the kinetic Boltzmann H-theorem. The nonextensive ingredients follow simply by modifying the molecular chaos hypothesis and by generalizing the local entropy expression according to Tsallis' argument. One important related result is that the proof of the H_q-theorem is possible (in accordance to the second law) only if the nonextensive parameter is positive definite, thereby fixing the lower bound assumed in equation (2). On the other hand, we know that high frequency vibrations in a collisionless electronic plasma may kinetically be described in a highly simplified manner, where electron-electron and electron-ion collisions are unimportant, in such a way that the collisional integral term in the Boltzmann equation may be neglected [1, 22]. In the linear approach, the distribution function of electrons is perturbed in first order, while the distribution function of ions can be considered as an invariable quantity. If $f_0(v)$ corresponds to the unperturbed homogeneous and time-independent stationary distribution, the resulting particle distribution function may be approximated as

$$f = f_0(v) + f_1(\mathbf{r}, \mathbf{v}, t), \quad f_1 \ll f_0, \tag{5}$$

where f_1 is the corresponding perturbation in the distribution function. The dynamical behavior of the plasma can be described by a combination of the linearized Vlasov and Poisson equations. One obtains [1, 22]

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{r}} = -\frac{e}{m}\nabla\phi_1.\frac{\partial f_0}{\partial \mathbf{v}}, \tag{6}$$

$$\nabla^2\phi_1 = -4\pi e\int f_1(\mathbf{r}, \mathbf{v}, t)d^3v, \tag{7}$$

where $\phi_1(\mathbf{r})$ is the first order correction in the electrostatic potential and e denotes the electronic charge. The set of coupled equations (6) and (7) may be worked out through the simplified derivation for electrostatic waves (longitudinal plasma waves), where the dispersion relation can be calculated either by taking the constraint of null permitivitty [2] or by using the mathematical techniques of integral transform originally developed by Landau [22]. In this case, the solutions of the above equations can be written as $f_1(\mathbf{r}, \mathbf{v}, t) = f_1(\mathbf{v})\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ and $\phi_1(\mathbf{r}, t) = \phi\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ provided that f_1 and ϕ satisfy the relations

$$(\mathbf{k}\cdot\mathbf{v} - \omega)f_1(\mathbf{v}) - \phi\mathbf{k}.\frac{\partial f_0}{\partial \mathbf{v}} = 0 \tag{8}$$

and

$$k^2\phi = 4\pi e\int f_1(\mathbf{v})d^3v. \tag{9}$$

Finally, combining these expressions we find that the dispersion relation between ω and $\mathbf{k}$ is given by

$$1 - \frac{4\pi e^2}{k^2 m}\int\frac{\mathbf{k}\cdot\partial f_0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}d^3v = 0. \tag{10}$$

Now, by considering the limit case of small wave number $k \ll k_D$ (from now on the subindex D stands for Debye quantities) we expand the integrand of (10) in power of

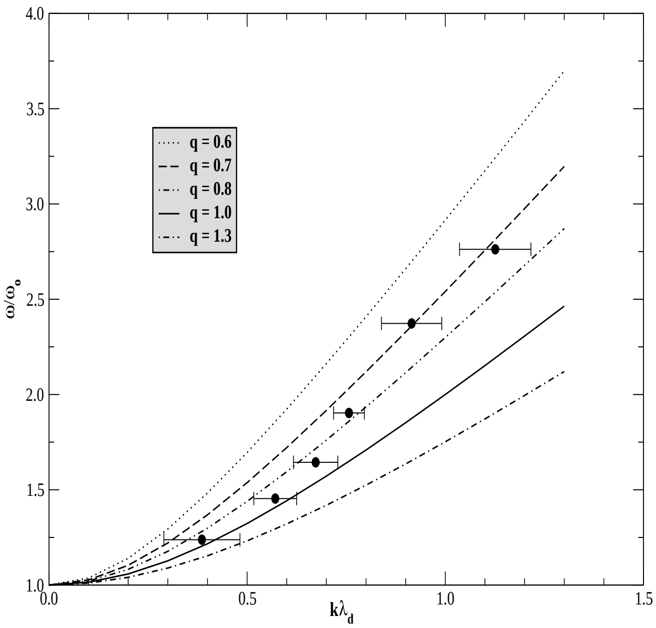

FIG. 1: Thermal dispersion relations for nonextensive velocity distribution. The selected values of q is showed in the picture. Note that nonextensive distribution with $q < 1$ is strongly suggested by these data.

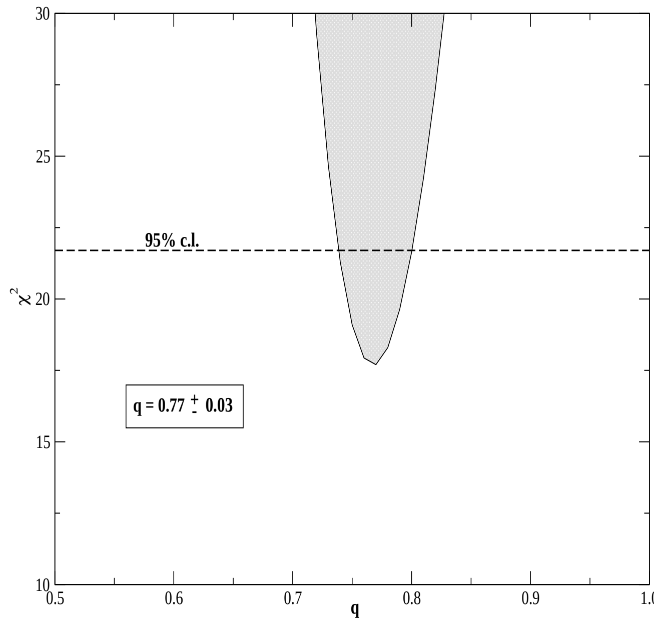

FIG. 2: $\chi^2 - q$ plane provided by the data of Van Hoven [14]. The horizontal line indicates the 95% confidence limit for the nonextensive parameter. From this analysis we find $q = 0.77 \pm 0.03$ (95% c.l.).

k and substitute the nonextensive stationary distribution given by (1) into the dispersion relation, one obtains (see reference [11] for more details)

$$W^2 = 1 + 3(\lambda_D k)^2 \left(\frac{2}{3q-1}\right), \tag{11}$$

where $W = \omega/\omega_0$, $\omega_0 = (\frac{4\pi n e^2}{m})^{1/2}$ is the natural oscillation plasma frequency and $\lambda_D = (k_B T/4\pi n e^2)^{1/2}$ is the electronic Debye-Hückel radius. As one may check, for the extensive limit $q = 1$, Eq. (11) reads

$$W^2 = 1 + 3(\lambda_D k)^2, \tag{12}$$

which is the standard Bohm-Gross relation [1]. In order to check quantitatively the validity of this new approach, as given by (11), we consider the experimental data set taken from Van Hoven [18]. These data, originally composed by 40 experimental points, was distributed into 6 bins by using the standard steps of data analysis (see, for instance, [23]). To find the confidence limits (c.l.) we use a χ^2 minimization neglecting the forbidden region by the H-theorem ($q < 0$) [21], i.e.,

$$\chi^2(k\lambda_D, q) = \sum_{i=1}^{5} \frac{[W(q) - W_{oi}]^2}{\sigma_i^2}, \tag{13}$$

where $W(q)$ is given by Eqs. (11) and (12) and W_{oi} is the experimental values of the ratio ω/ω_0 with errors σ_i of the i^{th} bin in the sample.

In Fig. 1 we show the binned data of the ratio W as a function of dimensionless parameter $k\lambda_D$ for some selected values of q. Thick solid line stands for the extensive Bohm-Gross result based on the Maxwellian distribution ($q = 1$). As can be seen, for values of $q > 1$ the curves increase less rapidly than in the Maxwellian case and clearly depart from the experimental points. In the complementary range, $0 < q < 1$, the curves increase more rapidly than in the Maxwellian case, showing a better agreement with the data. In Fig. 2 we display the result of our statistical analysis in the $\chi^2 - q$ plane. The horizontal dashed line corresponds to 95% confidence level (2σ). For these data the peak of the likelihood is located at $q = 0.77$ with the corresponding error interval of ± 0.03. Such a value suggests a power law behavior without thermal cutoff for the allowed values of velocities. These results clearly show that the new formalism discussed here may provide a better fit for these experimental data than does the standard extensive approach ($q = 1$). Moreover, the present discussion reinforces the interest for new experiments involving the dispersion relations in a collisionless electronic plasma in order to check more accurately the consistency of the results predicted by the nonextensive formalism.

In conclusion, we stress that the lack of a reasonable explanation to the physical meaning of the q parameter nowadays for long-range Hamiltonian systems points naturally to the following strategy: instead of paying attention to more formal results and mathematical ex-

tensions, it seems more important to constrain its value *via* a number of different set of experiments (see, for instance, [5]), in order to determine more clearly the reality of nonextensive effects. Actually, it has been shown [24] that the manganites (material that exhibit long-range interactions and fractal geometry) can be studied through the nonextensive statistic. In this regard, in contrast with long-range interactions of manganites, the present study shows that the role of long-range interactions does not seem to be a fundamental ingredient for a statistical analysis of thermal dispersion relation in plasma. In particular, we suspect that the extended kinetic theory can also be linked with statistical correlations through the velocities (before and after of a collision) of particles (see Refs. [20, 21]). Finally, we have showed that the Bohm-Gross dispersion relation which is a direct consequence of the Maxwellian distribution ($q = 1$) is strongly deprived by the experimental data, being ruled out by a considerably statistical margin. Our analysis indicates that the best fit for these particular data set occurs for a nonextensive distribution without thermal cutoff with $q \simeq 0.77$ in our convention for the q-exponential (see equation (2)). Note, however, that if the standard definition is considered (see for instance [16, 21]) these data set constrains the nonextensive parameter to the value $q = 1.23$ which is equivalent to a translation $q \to 2 - q$.

Acknowledgments

This work is supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq - Brasil) and CNPq (62.0053/01-1-PADCT III/Milenio). JSA is supported by CNPq (307860/2004-3)

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